Pseudohermitian Hamiltonians, time-reversal invariance and Kramers degeneracy

G. Scolarici*and L. Solombrino†
Dipartimento di Fisica dell'Università di Lecce
and INFN, Sezione di Lecce, I-73100 Lecce, Italy

July 30, 2002

Abstract

A necessary and sufficient condition in order that a (diagonalizable) pseudohermitian operator admits an antilinear symmetry $\mathfrak{T}$ such that $\mathfrak{T}^2 = -\mathbf{1}$ is proven. This result can be used as a quick test on the T-invariance properties of pseudohermitian Hamiltonians, and such test is indeed applied, as an example, to the Mashhoon-Papini Hamiltonian.

PACS numbers: 11.30.Er, 03.65.Ca, 03.65.Fd .

1 Introduction

Non Hermitian Hamiltonians are usually taken into account in order to describe dissipative systems or decay processes. In particular, in the last few years, a great attention has been devoted to the study of PT-symmetric quantum systems [1], whose Hamiltonians (though non Hermitian) possess real spectra, and in this context the interest rose on the class of pseudohermitian operators [2], i.e., those operators which satisfy

$$\eta H \eta^{-1} = H^{\dagger} \tag{1}$$

with $\eta = \eta^{\dagger}$ (of course, Hermiticity constitutes a particular case of pseudohermiticity, corresponding to $\eta = \mathbf{1}$).

When one considers diagonalizable operators with a discrete spectrum, one can prove that H is pseudohermitian if and only if its eigenvalues are either real or come in complex-conjugate pairs (with the same multiplicity) [3]; furthermore, this result has been generalized to all the (possibly non diagonalizable) matrix Hamiltonians [4], and to the class of all the PT-symmetric standard Hamiltonians having $\mathbf{R}$ as their configuration space [5] (which suggests that it may be valid under more general conditions).

*e-mail: scolarici@le.infn.it
†e-mail: solombrino@le.infn.it

Another physical reason for studying pseudohermitian operators is the remark that any T-invariant (diagonalizable) Hamiltonian must belong to their class [6]. The converse does not hold in general. Indeed, whereas one can prove that to any pseudohermitian operator is associated an antilinear symmetry [6][7], (in particular, at least, an involutory one), in general one cannot interpret it as the time-reversal operator T; furthermore, in case of fermionic systems, it is well known that

$$T^2 = -\mathbf{1}, \tag{2}$$

and the above-mentioned theorems do not ensure the existence of such a symmetry.

In order to deepen this point, we will prove in Sect. 2 that a Kramers-like degeneracy is a necessary and sufficient condition so as a diagonalizable pseudohermitian operator admits an antilinear symmetry which satisfies condition (2).

Next, as an example, we will apply in Sect. 3 the above result to the study of a non Hermitian Hamiltonian which has been recently proposed to interpret (by a T-violating spin-rotation coupling) a discrepancy between experimental and theoretical values of the muon's $g-2$ factor [8], and we will able to state precisely the parameters values associated with the T-violation.

2 A theorem on pseudohermitian operators

As in [3][6][7], we consider here only diagonalizable operators H with a discrete spectrum. Then, a complete biorthonormal eigenbasis $\{|\psi_{n,a}\rangle, |\phi_{n,a}\rangle\}$ exists [9], i.e., a basis such that

$$H\,|\psi_{n,a}\rangle = E_n\,|\psi_{n,a}\rangle\,, \qquad H^\dagger\,|\phi_{n,a}\rangle = E_n^*\,|\phi_{n,a}\rangle\,, \tag{3}$$

$$\langle\phi_{m,b}\,|\psi_{n,a}\rangle = \delta_{mn}\delta_{ab}, \tag{4}$$

$$\sum_n \sum_{a=1}^{d_n} |\psi_{n,a}\rangle\,\langle\phi_{n,a}| = \sum_n \sum_{a=1}^{d_n} |\phi_{n,a}\rangle\,\langle\psi_{n,a}| = \mathbf{1}, \tag{5}$$

where a, b are degeneracy labels and d_n denotes the degeneracy of E_n ; hence, the operator H can be written in the form

$$H = \sum_n \sum_{a=1}^{d_n} |\psi_{n,a}\rangle\,E_n\,\langle\phi_{n,a}|\,. \tag{6}$$

We can now state the following

Theorem. *Let H be diagonalizable operator with a discrete spectrum. Then, the following conditions are equivalent:*

i) an antilinear operator $\mathfrak{T}$ exists such that $[H, \mathfrak{T}] = 0$, with $\mathfrak{T}^2 = -1$;

ii) H is pseudohermitian and the degeneracy of its real eigenvalues is even.

Proof. Let us assume that condition $i)$ holds; then, H is pseudohermitian (see [6], Prop. 3 and Prop.1), hence its eigenvalues are either real or come in complex-conjugate pairs (with the same multiplicity). We will use in the following the subscript '$_0$' to denote real eigenvalues and the corresponding eigenvectors, and the subscript '$_\pm$' to denote the complex eigenvalues with positive or negative imaginary part, respectively, and the corresponding eigenvectors.

Let now $|\psi_{n_0,a}\rangle$ be an eigenvector of H; then, $\mathfrak{T}\,|\psi_{n_0,a}\rangle$ too is an eigenvector of H, corresponding to the same eigenvalue E_{n_0}, and linearly independent from $|\psi_{n_0,a}\rangle$. (Indeed, would be $\mathfrak{T}\,|\psi_{n_0,a}\rangle = \alpha\,|\psi_{n_0,a}\rangle$ for some $\alpha \in \mathbf{C}$, applying again $\mathfrak{T}$ to the previous relation we would obtain $|\psi_{n_0,a}\rangle = -\left|\alpha\right|^2 |\psi_{n_0,a}\rangle$, which is absurd.)

If $|\psi_{n_0,b}\rangle$ is another eigenvector of H, linearly independent from $|\psi_{n_0,a}\rangle$ and $\mathfrak{T}\,|\psi_{n_0,a}\rangle$, also $\mathfrak{T}\,|\psi_{n_0,b}\rangle$ is linearly independent from all three, otherwise, applying once again $\mathfrak{T}$ to the relation

$$\alpha\,|\psi_{n_0,a}\rangle + \beta\mathfrak{T}\,|\psi_{n_0.a}\rangle + \gamma\,|\psi_{n_0,b}\rangle + \delta\mathfrak{T}\,|\psi_{n_0,b}\rangle = 0$$

we could eliminate, for instance, $\mathfrak{T}\,|\psi_{n_0,b}\rangle$ obtaining so a linear dependence between $|\psi_{n_0,a}\rangle\,, \mathfrak{T}\,|\psi_{n_0,a}\rangle$ and $|\psi_{n_0,b}\rangle$, contrary to the previous hypothesis.

We can conclude, iterating this procedure, that d_{n_0} must be necessarily even.

Conversely, let condition $ii)$ hold, and let $\mathfrak{T}$ denote the following antilinear operator:

$$\begin{aligned} \mathfrak{T} \;&=\; \sum_{n_0}\sum_{a=1}^{d_{n_0}/2}\left(|\psi_{n_0,a}\rangle\,K\left\langle\phi_{n_0,a+d_{n_o/2}}\right| - \left|\psi_{n_0,a+d_{n_o/2}}\right\rangle K\left\langle\phi_{n_0,a}\right|\right) \qquad (7)\\ &\quad+\sum_{n_+,n_-,a}\left(\left|\psi_{n_-,a}\right\rangle K\left\langle\phi_{n_+,a}\right| - \left|\psi_{n_+,a}\right\rangle K\left\langle\phi_{n_-,a}\right|\right), \end{aligned}$$

where the operator K acts transforming each complex number on the right into its complex-conjugate. Then, one immediately obtains, by inspection, that $[H, \mathfrak{T}] = 0$ and $\mathfrak{T}^2 = -\mathbf{1}$. ■

The implication $i) \Longrightarrow ii)$ we proven above generalizes from various point of view the celebrated Kramers theorem on the degeneracy of any fermionic (Hermitian) Hamiltonian. Indeed, it applies to a larger class than that of the Hermitian operators (concerning their real eigenvalues only); moreover, it does not require a physical interpretation of the antilinear operator $\mathfrak{T}$ as a time-reversal operator. However, by an abuse of language, we will continue to denote as "*Kramers degeneracy*" this feature of pseudohermitian operators admitting a symmetry like $\mathfrak{T}$.

We stress once more that the Kramers degeneracy is a necessary but not a sufficient condition for the T-invariance.

3 Time-reversal violation in the spin-rotation coupling

On the basis of the previous discussions, we can quickly test the T-invariance properties of pseudohermitian Hamiltonians. To illustrate this point with an example, we chose a pseudohermitian Hamiltonian which has been recently introduced to interpret a discrepancy between experimental and standard model values of the muon's anomalous g factor.

In this model, a spin-rotation coupling, which involves small violations of the conservation of P and T, is considered. In particular, the spin-rotation effect described by Mashhoon [10] attributes an energy $-\frac{\hbar}{2}\overrightarrow{\omega}.\overrightarrow{\sigma}$ to a spin-$\frac{1}{2}$ particle in a frame rotating with angular velocity ω relative to an inertial frame. In the modified Mashhoon model [8] one assumes a different coupling of rotation to the right and left helicity states of the muon, $|\psi_+\rangle$ and $|\psi_-\rangle$. Hence, the total effective Hamiltonian is

$$H_{eff} = \begin{pmatrix} E & i(k_1\frac{\omega_2}{2} - \mu B) \\ -i(k_2\frac{\omega_2}{2} - \mu B) & E \end{pmatrix}, \tag{8}$$

where μ represents the total magnetic moment of the muon, B is the magnetic field, k_1, k_2 reflects the different coupling of rotation to the two helicity states.

Let us study in detail some properties of H_{eff} . A biorthonormal eigenbasis $\{|\psi_{1,2}\rangle, |\phi_{1,2}\rangle\}$ of H_{eff} is given by

$$\begin{aligned} |\psi_{1,2}\rangle &= \frac{1}{\sqrt{2}}[\pm i\chi^{\frac{1}{2}}|\psi_+\rangle + |\psi_-\rangle], \\ |\phi_{1,2}\rangle &= \frac{1}{\sqrt{2}}[\pm i\chi^{-\frac{1}{2}}|\psi_+\rangle + |\psi_-\rangle], \end{aligned}$$

where $\chi = \frac{k_1\omega_2 - 2\mu B}{k_2\omega_2 - 2\mu B}$. Its eigenvalues are

$$E_{1,2} = E \pm R,$$

where

$$R = \sqrt{(k_1\frac{\omega_2}{2} - \mu B)(k_2\frac{\omega_2}{2} - \mu B)},$$

therefore $E_{1,2}$ either are real or complex-conjugates. This peculiarity of its spectrum ensures us that H_{eff} is a pseudohermitian Hamiltonian [3], and indeed an Hermitian operator η exists which transform H_{eff} into $H^\dagger_{eff}$ (see Eq.(1)). In the case of real spectrum, for instance, η assumes the form [3][6]:

$$\eta = |\phi_1\rangle\langle\phi_1| + |\phi_2\rangle\langle\phi_2| = \begin{pmatrix} \frac{1}{\chi} & 0 \\ 0 & 1 \end{pmatrix}.$$

According to [8], a violation of (P and) T in H_{eff} would arise though $k_2 - k_1 \neq 0$. We can improve the discussion on the T-violating parameters values, by means of the Theorem in Sect. 2 . Indeed H_{eff} cannot be T-invariant for all the values of k_1 and k_2 which satisfy the condition

$$(k_1 \frac{\omega_2}{2} - \mu B)(k_2 \frac{\omega_2}{2} - \mu B) > 0 \tag{9}$$

since in this case H_{eff} has a real, non degenerate spectrum. (Note that by a suitable choice of B, condition (9) can be verified for all k_1, k_2.)

Let us indeed evaluate the (non unitary) evolution operator $U(t)$. This is given by [9]

$$\begin{aligned} U(t) &= |\psi_1\rangle e^{-iE_1 t}\langle\phi_1| + |\psi_2\rangle e^{-iE_2 t}\langle\phi_2| = \\ &\begin{pmatrix} e^{-iE_1 t} + e^{-iE_2 t} & i\chi^{\frac{1}{2}}(e^{-iE_1 t} - e^{-iE_2 t}) \\ -i\chi^{-\frac{1}{2}}(e^{-iE_1 t} - e^{-iE_2 t}) & e^{-iE_1 t} + e^{-iE_2 t} \end{pmatrix}. \end{aligned} \tag{10}$$

Then, assuming the initial condition $|\psi(0)\rangle = |\psi_-\rangle$, the muon's state at the time t is

$$|\psi(t)\rangle = \frac{1}{2}[i\chi^{\frac{1}{2}}(e^{-iE_1 t} - e^{-iE_2 t})|\psi_+\rangle + (e^{-iE_1 t} + e^{-iE_2 t})|\psi_-\rangle].$$

The spin-flip probability is therefore

$$P(t)_{\psi_- \to \psi_+} = |\langle\psi_+|\psi(t)\rangle|^2 = \frac{\chi}{2}[1 - \cos 2Rt], \tag{11}$$

which agrees with the analogous calculation in [8] (where, however, also the width Γ of the muon is taken into account).

Note that the above probability do not depend on the sign of the time; this feature occurs whenever (in a two level system) a transition probability between orthogonal states is considered, and disappears when a different choice of the states is made. Actually, evaluating for instance the transition probability between the states $|\psi_-\rangle$ and $|\varphi\rangle = \frac{1}{\sqrt{2}}[|\psi_+\rangle - |\psi_-\rangle]$ one obtains

$$P(t)_{\psi_- \to \varphi} = |\langle\varphi|\psi(t)\rangle|^2 = \frac{1}{2}(\cos Rt + \chi^{\frac{1}{2}} \sin Rt)^2, \tag{12}$$

and $P(t)_{\psi_- \to \varphi} - P(-t)_{\psi_- \to \varphi} = \chi^{\frac{1}{2}} \sin 2Rt \neq 0$, which explicitly shows that H_{eff} is a T-violating Hamiltonian (even if $k_1 = k_2$), in agreement with our Theorem.

.

References

[1] C. M. Bender and S. Boettcher, *Phys. Rev. Lett.* **80**, 5243 (1998);

F. M. Fernandez, R. Guardiola, J. Ros and M. Znojil, *J. Phys A: Math. Gen.* **31**, 10105 (1998);

F. Cannata, G. Junker and J. Trost, *Phys. Lett.* **A 246**, 219 (1998);

C. M. Bender, S. Boettcher and P. N. Meisenger, *J. Math. Phys.* **40**, 2201 (1999).

[2] T. D. Lee and G. C. Wick, *Nucl. Phys.* **B9**, 209 (1969).

[3] A. Mostafazadeh, *J. Math. Phys.* **43**, 205 (2002).

[4] A. Mostafazadeh, "Pseudo-Hermiticity for a class of Nondiagonalizable Hamiltonians", arXiv: math-ph/0207009 (2002).

[5] A. Mostafazadeh, "On the Pseudo-Hermiticity of General PT-Symmetric Standard Hamiltonians in One Dimension", arXiv: math-ph/0204013 (2002).

[6] L. Solombrino, "Weak pseudo-Hermiticity and antilinear commutant" (arXiv: quant-ph/0203101), *J. Math. Phys.*, to appear.

[7] A. Mostafazadeh, "Pseudo-Hermiticity versus PT-Symmetry III: Equivalence of pseudo-Hermiticity and the presence of antilinear symmetries", arXiv: math-ph/0203005 (2002).

[8] G. Papini, *Phys. Rev.* **D65**, 077901 (2002).

[9] F. H. M. Faisal and J. V. Moloney, *J. Phys. B: At. Mol. Phys.* **14**, 3603 (1981).

[10] B. Mashhoon, *Phys. Rev. Lett.* **61**, 2639 (1988).